Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Approximation Theory 159 (2009) 167–179
Journal of Approximation Theory
www.elsevier.com/locate/jat

Cesàro means of Jacobi expansions on the parabolic biangle

W. zu Castell[a,*], F. Filbir[a], Y. Xu[b]

[a] Institute of Biomathematics and Biometry, Helmholtz Zentrum München, German Research Center for Environmental Health, Ingolstädter Landstraße 1, 85764 Neuherberg, Germany
[b] Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA

Received 15 May 2008; accepted 8 January 2009
Available online 15 January 2009

Communicated by C.K. Chui and H.N. Mhaskar

Dedicated to the memory of G.G. Lorentz

Abstract

We study Cesàro (C, δ) means for two-variable Jacobi polynomials on the parabolic biangle $B = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1^2 \le x_2 \le 1\}$. Using the product formula derived by Koornwinder and Schwartz for this polynomial system, the Cesàro operator can be interpreted as a convolution operator. We then show that the Cesàro (C, δ) means of the orthogonal expansion on the biangle are uniformly bounded if $\delta > \alpha + \beta + 1$, $\alpha \ge \beta \ge 0$. Furthermore, for $\delta \ge \alpha + 2\beta + \frac{3}{2}$ the means define positive linear operators.

Keywords: Orthogonal expansion; Cesàro summability; Parabolic biangle; Two-variable orthogonal polynomials; Positive linear operators; Convolution operators

1. Introduction

Multivariate analogs of classical orthogonal polynomials are of great interest in many areas of applied analysis and approximation. Although, there is a beautiful theory of orthogonal polynomials in one variable (cf. [10]), it is much harder to develop a picture as comprehensive as in the univariate case for orthogonal polynomials in several variables (cf. [3]). Among classical orthogonal polynomials in one variable the family of Jacobi polynomials plays a special role. Their importance is partly due to the fact that there are connections of these polynomials to

* Corresponding author.
E-mail addresses: castell@helmholtz-muenchen.de, castell@gsf.de (W. zu Castell), filbir@helmholtz-muenchen.de, filbir@gsf.de (F. Filbir), yuan@uoregon.edu (Y. Xu).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2009.01.002

group representations and the eigensystem of the Laplace–Beltrami operator for certain compact symmetric spaces. In the two-dimensional setting, Koornwinder [5,6] discusses seven classes of Jacobi polynomials in two variables, derived by either expressions in terms of univariate Jacobi polynomials, studying spherical functions on homogeneous spaces of rank 2, or using quadratic transformations of known examples of Jacobi polynomials, as well as analytic continuation with respect to some parameters. Among these classes, there are bivariate Jacobi polynomials on the biangle. The *parabolic biangle* is the closed subset of $\mathbb{R}^2$ defined by

$$B = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1^2 \le x_2 \le 1\}.$$

The system of bivariate orthogonal polynomials defined on the set B has first been introduced by Agahanov [1]. The polynomials can be explicitly expressed in terms of the univariate Jacobi polynomials. In order to provide the explicit formulae let us recall some basic facts for Jacobi polynomials.

For $\alpha, \beta > -1$ and $n \in \mathbb{N}_0$ the Jacobi polynomials are defined as

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right),$$

where $F(a, b; c; x)$ denotes the Gaussian hypergeometric function. Given the weight function $w^{(\alpha,\beta)}$ defined as

$$w^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^\alpha(1+x)^\beta, \quad x \in (-1, 1),$$

the Jacobi polynomials satisfy the following orthogonality relation:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)\, \mathrm{d}x = \delta_{n,m} h_n^{(\alpha,\beta)},$$

where

$$h_n^{(\alpha,\beta)} = \frac{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+1)}{n!(\alpha+\beta+2)_n(\alpha+\beta+2n+1)_n}. \tag{1.1}$$

We are now ready to define the orthogonal polynomials system on B. For $\alpha, \beta > -1$ and $n \ge k \ge 0$ let

$$P_{n,k}^{\alpha,\beta}(\mathbf{x}) = P_{n-k}^{(\alpha-\frac{1}{2},\beta+k)}(2x_2-1) \cdot x_2^{\frac{k}{2}} P_k^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(x_1 x_2^{-\frac{1}{2}}), \quad \mathbf{x} = (x_1, x_2) \in B. \tag{1.2}$$

Note that the polynomial is of degree k in x_1 and of degree $n-k$ in x_2 and thus, of total degree n. Furthermore,

$$P_{n,k}^{\alpha,\beta}(\mathbf{e}) = \frac{\left(\alpha+\frac{1}{2}\right)_{n-k}\left(\beta+\frac{1}{2}\right)_k}{(n-k)!k!},$$

where $\mathbf{e} = (1, 1) \in B$. Moreover, for α, β fixed these polynomials are orthogonal with respect to the weight function

$$W^{\alpha,\beta}(\mathbf{x}) = W_B^{\alpha,\beta}(\mathbf{x}) = \frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}(1-x_2)^{\alpha-\frac{1}{2}}(x_2-x_1^2)^{\beta-\frac{1}{2}},$$
$$\mathbf{x} = (x_1, x_2) \in B. \tag{1.3}$$

Thus, for the L^2-norm of the polynomials $P_{n,k}^{\alpha,\beta}$ we obtain the expression

$$\begin{aligned}\left(g_{n,k}^{\alpha,\beta}\right)^{-1} &= \int_B |P_{n,k}^{\alpha,\beta}(\mathbf{x})|^2\, W^{\alpha,\beta}(\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &= h_{n-k}^{(\alpha-\frac{1}{2},\beta+k)} h_k^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}.\end{aligned} \tag{1.4}$$

To keep the notation simple we write $L^p(B)$ for the Lebesgue spaces $L^p(B, W^{\alpha,\beta})$, $1 \le p \le \infty$, where for $p = \infty$ the space $L^\infty(B)$ is understood to be the space of continuous functions on B with the supremum norm.

We want to study Cesàro means of orthogonal expansions with respect to the system $\{P_{n,k}^{\alpha,\beta} : 0 \le k \le n\}$, $n \in \mathbb{N}_0$, on B.

For $\delta \ge 0$, the *Cesàro* (C, δ) *means* s_n^δ of a sequence $(c_n)_{n=0}^\infty$ are defined as

$$s_n^\delta = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta-1}{n-k} s_k = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} c_k, \tag{1.5}$$

where s_n denotes the nth partial sum $\sum_{k=0}^n c_k$. The sequence $(s_n)_{n=0}^\infty$ is called (C, δ) *summable* if s_n^δ converges for $n \to \infty$. For simplicity, we set $A_n^\delta = \binom{n+\delta}{n}$.

The study of Cesàro means for orthogonal expansions has a long history. It is worth noting that in the multivariate setting, there is a strong influence of the domain of definition, which is up until now not fully understood. For example, the Cesàro means for the *Chebyshev weight function* $1/\sqrt{(1-x^2)(1-y^2)}$ converge uniformly on the unit square for all $\delta > 0$. On the unit ball, there is in contrast a *critical index* for the weight $1/\sqrt{1-\|x\|^2}$, which means, uniform convergence does not hold if $\delta \le \frac{d-1}{2}$. The same is true for the standard simplex (cf. [2,9] for further details). The parabolic biangle is in some sense an intermediate region having both, a smooth curved boundary and singularities. As we will show below, for $1/\sqrt{(1-x_2)(x_2-x_1^2)}$ on this region uniform convergence holds if $\delta > 1$.

In the following section we will introduce the Cesàro means for the orthogonal expansion on the biangle. Since one crucial point for our following discussion is the fact that the Cesàro means can be interpreted as a convolution operator, we recall some facts on the convolution structure on the parabolic biangle. In Section 3 our main results are stated, while the proof of Theorem 3.2, our main result, is given in the last section. Throughout the paper we will use the letter c for a generic constant which does not have to be the same in every occurrence.

2. Cesàro means on the biangle

The Fourier coefficient of a function $f \in L^1(B)$ with respect to the orthogonal system $\{P_{n,k}^{\alpha,\beta} : 0 \le k \le n\}$, $n \in \mathbb{N}_0$, is given by

$$\widehat{f}(n,k) = \int_B f(\mathbf{y})\, P_{n,k}^{\alpha,\beta}(\mathbf{y})\, W^{\alpha,\beta}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{2.1}$$

where $0 \le k \le n$, $n \in \mathbb{N}_0$.

The projection of f onto the space of polynomials of degree n is given by

$$\mathcal{P}_n f(\mathbf{x}) = \sum_{k=0}^n \widehat{f}(n,k)\, P_{n,k}^{\alpha,\beta}(\mathbf{x}) g_{n,k}^{\alpha,\beta} = \int_B f(\mathbf{x}) P_n(\mathbf{x},\mathbf{y})\, W^{\alpha,\beta}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{2.2}$$

with kernel

$$P_n(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{n} P_{n,k}^{\alpha,\beta}(\mathbf{x}) P_{n,k}^{\alpha,\beta}(\mathbf{y}) g_{n,k}^{\alpha,\beta}.$$

We want to study approximate expansions of functions using summability methods. Let us therefore recall the definition of Cesàro means.

The Cesàro (C,δ) means of the expansion (2.2) are given by

$$\mathcal{S}_n^\delta f(\mathbf{x}) = \sum_{k=0}^{n} \frac{A_{n-k}^\delta}{A_n^\delta} \mathcal{P}_k f(\mathbf{x}) = \int_B f(\mathbf{y}) \mathcal{K}_n^\delta(\mathbf{x},\mathbf{y})\, W^{\alpha,\beta}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{2.3}$$

where the summability kernel is given by

$$\begin{aligned}\mathcal{K}_n^\delta(\mathbf{x},\mathbf{y}) &= \sum_{k=0}^{n} \frac{A_{n-k}^\delta}{A_n^\delta} P_k(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{A_{n-k}^\delta}{A_n^\delta} P_{k,l}^{\alpha,\beta}(\mathbf{x}) P_{k,l}^{\alpha,\beta}(\mathbf{y}) g_{k,l}^{\alpha,\beta} \\ &= \sum_{k=0}^{n} \frac{A_{n-k}^{\delta-1}}{A_n^\delta} \mathcal{K}_k(\mathbf{x},\mathbf{y}). \end{aligned} \tag{2.4}$$

Here and for the remaining part of the paper we simply write $\mathcal{S}_n$ for $\mathcal{S}_n^0$ and $\mathcal{K}_n$ for $\mathcal{K}_n^0$. Now, a key observation is that the operators $\mathcal{P}_n$ and $\mathcal{S}_n^\delta$ can be written as convolution operators. In order to see how this works, we need to outline some facts concerning the convolution structure on B associated with the polynomial system $\{P_{n,k}^{\alpha,\beta} : 0 \le k \le n\}$.

Convolution structures are commonly based on product formulae. For the polynomials $P_{n,k}^{\alpha,\beta}$ such a construction was given by Koornwinder and Schwartz [7]. We need some notation. For $\mathbf{x} = (x_1, x_2)$ with $|x_1|, |x_2| \in [0,1]$, $0 \le r \le 1$, and $\psi, \psi_j \in [0,\pi]$, $j = 1,2,3$, define

$$\begin{aligned} D(\mathbf{x}; r,\psi) &= x_1x_2 + (1-x_1^2)^{1/2}(1-x_2^2)^{1/2}\, r\cos\psi, \\ E(\mathbf{x}; r,\psi) &= (x_1^2x_2^2 + (1-x_1^2)(1-x_2^2)r^2 + 2x_1x_2(1-x_1^2)^{1/2}(1-x_2^2)^{1/2}\, r\cos\psi)^{1/2}, \end{aligned}$$

and, setting $\mathbf{y} = (y_1, y_2)$,

$$F(\mathbf{x},\mathbf{y}; r,\psi_1,\psi_2,\psi_3) = E(\mathbf{x}; r,\psi_1) D\left[\frac{D(\mathbf{x}; r,\psi_1)}{E(\mathbf{x}; r,\psi_1)}, D\left(\frac{x_1}{x_2},\frac{y_1}{y_2}; 1,\psi_2\right); 1,\psi_3\right],$$

where $\mathbf{x},\mathbf{y} \ne 0$. Moreover, we define the following measures

$$\mathrm{d}m^{\alpha,\beta}(r,\psi) = \frac{2\Gamma(\alpha+1/2)\Gamma(\beta+1/2)}{\Gamma(\alpha-\beta)\Gamma(\beta+1/2)\Gamma(\beta)\Gamma(1/2)}(1-r^2)^{\alpha-\beta-1} r^{2\beta}\sin^{2\beta-1}\psi\,\mathrm{d}r\,\mathrm{d}\psi$$

and

$$\begin{aligned}\mathrm{d}\mu^{\alpha,\beta}(r,\psi_1,\psi_2,\psi_3) &= c_{\alpha,\beta}(1-r^2)^{\alpha-\beta-3/2} r^{2\beta} \\ &\quad\times \sin^{2\beta-1}\psi_2 \sin^{2\beta-1}\psi_3 \sin^{2\beta}\psi_1\,\mathrm{d}r\,\mathrm{d}\psi_1\,\mathrm{d}\psi_2\,\mathrm{d}\psi_3,\end{aligned}$$

where $c_{\alpha,\beta}$ is a constant so that $\mathrm{d}\mu^{\alpha,\beta}$ is a probability measure.

Koornwinder and Schwartz [7] proved the following product formula for the orthogonal polynomials on the parabolic biangle.

Theorem 2.1. *Let $\alpha \geq \beta \geq 0$. Assume that $0 \leq |x_1| \leq x_2 \leq 1$ and $0 \leq |y_1| \leq y_2 \leq 1$. Then if $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in B \setminus \{\mathbf{0}\}$, we have that*

$$\begin{aligned} &P_{n,k}^{\alpha,\beta}(x_1, x_2^2)\, P_{n,k}^{\alpha,\beta}(y_1, y_2^2) \\ &\quad = P_{n,k}^{\alpha,\beta}(\mathbf{e}) \int_{[0,1]\times[0,\pi]^3} P_{n,k}^{\alpha,\beta}(E^2(\mathbf{x}; r, \psi), F(\mathbf{x}, \mathbf{y}; r, \psi_1, \psi_2, \psi_3))\mathrm{d}\mu^{\alpha,\beta}(r, \psi_1, \psi_2, \psi_3). \end{aligned}$$

If $0 \leq |x_1| \leq x_2 \leq 1$ and $\mathbf{y} = \mathbf{0}$ we have that

$$\begin{aligned} &P_{n,k}^{\alpha,\beta}(x_1, x_2^2)\, P_{n,k}^{\alpha,\beta}(\mathbf{0}) \\ &\quad = P_{n,k}^{\alpha,\beta}(\mathbf{e}) \int_{[0,1]\times[0,\pi]^3} P_{n,k}^{\alpha,\beta}(E^2(\mathbf{x}; r, \psi), D(\mathbf{x}; r, \psi_1))\mathrm{d}m^{\alpha,\beta+1/2}(r, \psi_1). \end{aligned}$$

For our aim we do not need the product formula in its explicit form, rather than its existence, which gives rise to the convolution structure on the parabolic biangle. Thus, let us restate the product formula in a convenient form:

$$P_{n,k}^{\alpha,\beta}(\mathbf{x})\, P_{n,k}^{\alpha,\beta}(\mathbf{y}) = P_{n,k}^{\alpha,\beta}(\mathbf{e}) \int_B P_{n,k}^{\alpha,\beta}(\mathbf{z})\, \mathrm{d}\omega_{\mathbf{x},\mathbf{y}}(\mathbf{z}), \tag{2.5}$$

where $\mathrm{d}\omega_{\mathbf{x},\mathbf{y}}(\mathbf{z})$ is a probability measure.

Formula (2.5) gives rise to a generalized translation operator on B in the following way. For $f \in C(B)$ we define

$$T_{\mathbf{x}} f(\mathbf{y}) = \int_B f(\mathbf{z})\, \mathrm{d}\omega_{\mathbf{x},\mathbf{y}}(\mathbf{z}), \quad \mathbf{x}, \mathbf{y} \in B.$$

It can be shown that $T_{\mathbf{x}}$ extends to a bounded linear operator on $L^p(B)$ with $\|T_{\mathbf{x}}\|_p \leq 1, \mathbf{x} \in B$, for every $1 \leq p \leq \infty$. The convolution of functions $f, g \in L^1(B)$ is then defined as

$$f * g(\mathbf{x}) = \int_B f(\mathbf{y}) T_{\mathbf{x}} g(\mathbf{y})\, W^{\alpha,\beta}(\mathbf{y})\, \mathrm{d}\mathbf{y}, \quad \mathbf{x} \in B. \tag{2.6}$$

This convolution product is associative and commutative. Moreover, the following estimate holds true for all $f, g \in L^p(B)$:

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p, \quad 1 \leq p \leq \infty. \tag{2.7}$$

In view of formulae (2.5) and (2.6) it becomes obvious that the operator $\mathcal{S}_n^\delta$ can be written as convolution operator. To be precise, we have the following result.

Proposition 2.1. *If $\alpha \geq \beta \geq 0$, $\delta > 0$, and $\mathbf{x} \in B$, then for every $f \in L^p(B)$ we have that*

$$\mathcal{S}_n^\delta f(\mathbf{x}) = K_n^\delta * f(\mathbf{x}), \tag{2.8}$$

where $K_n^\delta(\mathbf{x}) = \mathcal{K}_n^\delta(\mathbf{x}, \mathbf{e})$ and $\mathbf{e} = (1, 1)$.

Proof. From Eqs. (2.3) and (2.4) we obtain, using the product formula (2.5), that

$$\begin{aligned} \mathcal{S}_n^\delta f(\mathbf{x}) &= \int_B f(\mathbf{y}) \sum_{k=0}^n \sum_{l=0}^k \frac{A_{n-k}^\delta}{A_n^\delta} P_{k,l}^{\alpha,\beta}(\mathbf{e}) T_{\mathbf{x}} P_{k,l}^{\alpha,\beta}(\mathbf{y})\, W^{\alpha,\beta}(\mathbf{y})\, \mathrm{d}\mathbf{y} \\ &= K_n^\delta * f(\mathbf{x}). \quad \square \end{aligned}$$

Now from inequality (2.7) an estimate for the operator norm of $\mathcal{S}_n^\delta$ follows, i.e., $\|\mathcal{S}_n^\delta\|_p \le \|K_n^\delta\|_1$, $1 \le p \le \infty$. Notice that $K_n^\delta(\mathbf{x})$ is the kernel $\mathcal{K}_n^\delta(\mathbf{x},\mathbf{y})$ when $\mathbf{y} = \mathbf{e}$. The uniform convergence of the (C,δ) means on B is reduced to convergence at a point. We state this formerly as a corollary.

Corollary 2.2. *If $\alpha \ge \beta \ge 0$, $\delta > 0$, and $f \in L^p(B)$, the Cesàro (C,δ) means $\mathcal{S}_n^\delta f$ converge uniformly on B if $\mathcal{S}_n^\delta f$ converges at the point* $\mathbf{e}$*, which holds, in turn, if*

$$\|K_n^\delta\|_1 = \int_B |K_n^\delta(\mathbf{x})| W^{(\alpha,\beta)}(\mathbf{x}) \mathrm{d}\mathbf{x} \le c < \infty \tag{2.9}$$

uniformly in n.

3. Summability of Cesàro expansions on the biangle

As shown in Corollary 2.2, it is sufficient to establish (2.9). For this purpose, it is essential to derive a closed formula for the kernel K_n^δ, which we state below.

Theorem 3.1. *For $\alpha, \beta > -\frac{1}{2}$ and $\mathbf{x} \in B$ we have that*

$$\begin{aligned} K_n(\mathbf{x}) = {} & P_n^{(\alpha+\beta+\frac{1}{2},\beta)}(1) \int_{-1}^1 P_n^{(\alpha+\beta+\frac{1}{2},\beta)} \left(\frac{1}{2}(1+t)^2 + (1-t^2)x_1 + \frac{1}{2}(1-t)^2 x_2 - 1 \right) \\ & \times w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)\, \mathrm{d}t. \end{aligned} \tag{3.1}$$

Proof. We derive this formula from the addition formula for Jacobi polynomials $P_n^{(\alpha,\beta)}$ established by Koornwinder (cf. [5, (4.14)]).

$$\begin{aligned} & P_n^{(\alpha,\beta)}\Big(\frac{1}{2}(1+\xi)(1+\eta) + \frac{1}{2}(1-\xi)(1-\eta)r^2 + (1-\xi^2)^{1/2}(1-\eta^2)^{1/2}\, r\cos\psi - 1\Big) \\ & = \sum_{k=0}^n \sum_{l=0}^k a_{n,k,l}^{(\alpha,\beta)} (1-\xi)^{(k+l)/2}(1+\xi)^{(k-l)/2} P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\xi) \\ & \quad \times (1-\eta)^{(k+l)/2}(1+\eta)^{(k-l)/2} P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\eta) \\ & \quad \times P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\, r^{k-l}\, P_{k-l}^{(\beta-1/2,\beta-1/2)}(\cos\psi), \end{aligned} \tag{3.2}$$

where the coefficients $a_{n,k,l}^{(\alpha,\beta)}$ are given by

$$\begin{aligned} a_{n,k,l}^{(\alpha,\beta)} = {} & (k+l+\alpha)(k-l+\beta) \\ & \times \frac{(n+\alpha+\beta+1)_k (2\beta+1)_{k-l} (n-l+\beta+1)_l (n-k)!}{2^{2k}(k+\alpha)(\frac{1}{2}(k-l)+\beta)(\beta+1)_k (k+\alpha+1)_{n-k+l}(\beta+1/2)_{k-l}}. \end{aligned} \tag{3.3}$$

By definition (2.4) we have that

$$K_n(\mathbf{x},\mathbf{e}) = \sum_{k=0}^n \sum_{l=0}^k P_{k,l}^{\alpha,\beta}(\mathbf{x}) P_{k,l}^{\alpha,\beta}(\mathbf{e}) g_{k,l}^{\alpha,\beta}.$$

From the fact that $P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!$, it follows that

$$P_{k,l}^{\alpha,\beta}(\mathbf{e}) = P_{k-l}^{(\alpha-\frac{1}{2},\beta+k)}(1) P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(1) = \frac{(\alpha+\frac{1}{2})_{k-l}(\beta+\frac{1}{2})_l}{(k-l)!\,l!}.$$

In the addition formula (3.2), set $\xi = \eta = t$, $r = \sqrt{x_2}$, $r\cos\psi = x_1$, and replace α by $\alpha+\beta+1/2$ to obtain

$$P_n^{(\alpha+\beta+\frac{1}{2},\beta)}\Big(\frac{1}{2}(1+t)^2 + (1-t^2)x_1 + \frac{1}{2}(1-t)^2 x_2 - 1\Big)$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{k} a_{n,k,l}^{(\alpha+\beta+\frac{1}{2},\beta)}(1-t)^{k+l}(1+t)^{k-l}\Big[P_{n-k}^{(\alpha+\beta+k+l+\frac{1}{2},\beta+k-l)}(t)\Big]^2$$
$$\times P_l^{(\alpha-\frac{1}{2},\beta+k-l)}(2x_2-1)x_2^{\frac{k-l}{2}} P_{k-l}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}\Big(\frac{x_1}{\sqrt{x_2}}\Big).$$

The product of the last three terms in the sum is precisely $P_{k,k-l}^{\alpha,\beta}$. Integrating the above equation with respect to $w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)\,\mathrm{d}t$ gives

$$\int_{-1}^{1} P_n^{(\alpha+\beta+\frac{1}{2},\beta)}\Big(\frac{1}{2}(1+t)^2 + (1-t^2)x_1 + \frac{1}{2}(1-t)^2 x_2 - 1\Big) w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)\,\mathrm{d}t$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{k} a_{n,k,l}^{(\alpha+\beta+\frac{1}{2},\beta)} b_{n,k,l}^{(\alpha,\beta)} P_{k,k-l}^{\alpha,\beta}(\mathbf{x}),$$

where

$$b_{n,k,l}^{(\alpha,\beta)} = \int_{-1}^{1}\Big[P_{n-k}^{(\alpha+\beta+k+l+\frac{1}{2},\beta+k-l)}(t)\Big]^2(1-t)^{k+l}(1+t)^{k-l} w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)\,\mathrm{d}t.$$

Using (1.1) and the explicit formula of $a_{n,k,l}^{(\alpha+\beta+\frac{1}{2},\beta)}$ it can be verified that

$$a_{n,k,l}^{(\alpha+\beta+\frac{1}{2},\beta)} b_{n,k,l}^{(\alpha,\beta)} = g_{k-l,k}^{\alpha,\beta} P_{k,k-l}^{\alpha,\beta}(\mathbf{e})[P_n^{(\alpha+\beta+\frac{1}{2},\beta)}(1)]^{-1} h_n^{(\alpha+\beta+\frac{1}{2},\beta)}.$$

This proves the stated formula. □

We will use the abbreviation $z(\mathbf{x};t)$ for the argument of the univariate Jacobi polynomial in Eq. (3.1), i.e.,

$$z(\mathbf{x};t) = \frac{1}{2}(1+t)^2 + (1-t^2)x_1 + \frac{1}{2}(1-t)^2 x_2 - 1, \quad \mathbf{x} = (x_1,x_2) \in B.$$

Using Theorem 3.1 we obtain the following form of the Cesàro kernel.

Corollary 3.1. *Let $\alpha \ge \beta \ge 0$ and $\delta > 0$. Then for all $\mathbf{x} \in B$ and $f \in L^p(B)$, $1 \le p \le \infty$, we have that*

$$K_n^\delta(\mathbf{x}) = \sum_{k=0}^{n} \frac{A_{n-k}^{\delta-1}}{A_n^\delta} h_k^{(\alpha+\beta+\frac{1}{2},\beta)} P_k^{(\alpha+\beta+\frac{1}{2},\beta)}(1) \int_{-1}^{1} P_k^{(\alpha+\beta+\frac{1}{2},\beta)}(z(\mathbf{x};t))\, w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)\,\mathrm{d}t$$
$$= \frac{A_n^{\delta-1}}{A_n^\delta}\int_{-1}^{1} k_n^{\delta-1}(z(\mathbf{x};t))\, w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)\,\mathrm{d}t, \tag{3.4}$$

where k_n^δ is the univariate Cesàro (C,δ) kernel,

$$k_n^\delta(t) = \sum_{k=0}^{n} \frac{A_{n-k}^{\delta}}{A_n^\delta} h_k^{(\alpha+\beta+\frac{1}{2},\beta)} P_k^{(\alpha+\beta+\frac{1}{2},\beta)}(1) P_k^{(\alpha+\beta+\frac{1}{2},\beta)}(t).$$

Eq. (3.4) allows us to establish (2.9) by working with the univariate kernel k_n^δ. This leads to our main result in this paper.

Theorem 3.2. *Let $\alpha \ge \beta \ge 0$. The (C, δ) means of the orthogonal expansions with respect to $W_B^{\alpha,\beta}$ converge uniformly to f on B for every continuous function f if $\delta > \alpha + \beta + 1$.*

The proof of Theorem 3.2 will be stated in Section 4. The proof involves sharp estimates on various pieces, which indicates that the order $\delta > \alpha + \beta + 1$ is sharp. In other words, we conjecture that the convergence fails if $\delta \le \alpha + \beta + 1$.

The compact formula (3.4) also allows us to deduce positivity of the (C, δ) means on B from its counterpart of univariate Jacobi expansions.

Theorem 3.3. *Let $\alpha \ge \beta \ge 0$. The $(C, \alpha + 2\beta + 3/2)$ means of the orthogonal expansions with respect to $W_B^{\alpha,\beta}$ define positive linear operators.*

Proof. In [4] it is proved that the $(C, \mu + \nu + 2)$ means of the univariate Jacobi expansion

$$\sum_{n=0}^{\infty} [h_n^{\mu,\nu}]^{-1} P_n^{(\mu,\nu)}(1) P_n^{(\mu,\nu)}(x)$$

are nonnegative for $-1 \le x \le 1$, $\mu \ge \nu \ge -1/2$. By Corollary 3.1, the (C, δ) means of $K_n(\mathbf{x})$ are the integrals of the $(C, \delta - 1)$ means of the Jacobi expansions for $w^{(\alpha+\beta+\frac{1}{2},\beta)}$, which shows that $K_n^\delta(\mathbf{x})$ is nonnegative for $\delta = \alpha + 2\beta + 3/2$. By the product formula, $\mathcal{K}_n^\delta(\mathbf{x}, \mathbf{y}) = T_{\mathbf{x}} \mathcal{K}_n(\mathbf{e}, \mathbf{y})$, and $T_{\mathbf{x}}$ is an integral against a nonnegative measure, it follows that $\mathcal{K}_n^\delta(\mathbf{x}, \mathbf{y}) \ge 0$ for $\delta = \alpha + 2\beta + 3/2$. □

We note that if the (C, δ_0) means are nonnegative, then the (C, δ) means are nonnegative for all $\delta \ge \delta_0$. The positivity implies $\|K_n^\delta\|_1 = 1$, hence, convergence of the means. However, since convergence of the (C, δ) means implies the convergence of the (C, δ') means for $\delta' > \delta$, the convergence also follows from Theorem 3.2.

4. Proof of Theorem 3.2

We start with a result in [10, p. 261, (9.4.13)] and its extension in [8] given in the following lemma.

Lemma 4.1. *Let $k_n^\delta(w^{(\xi,\eta)}; u)$, $u \in [-1, 1]$, denote the kernel for the univariate Cesàro (C, δ) means of the Jacobi expansion with respect to the weight function $w^{(\xi,\eta)}$ on $[-1, 1]$. Then for any $\xi, \eta > -1$ such that $\xi + \eta + \delta + 3 > 0$ we have that*

$$k_n^\delta(w^{(\xi,\eta)}; t) = \sum_{j=0}^{J} b_j(\xi, \eta, \delta, n) P_n^{(\xi+\delta+j+1,\eta)}(t) + G_n^\delta(t),$$

where J is a fixed integer and

$$G_n^\delta(t) = \sum_{j=J+1}^{\infty} d_j(\xi, \eta, \delta, n) k_n^{\delta+j}(w^{(\xi,\eta)}, 1, t).$$

Moreover, the coefficients satisfy the inequalities

$$|b_j(\xi, \eta, \delta, n)| \le c n^{\xi+1-\delta-j} \quad \text{and} \quad |d_j(\xi, \eta, \delta, n)| \le c j^{-\xi-\eta-\delta-4}.$$

Furthermore, we will need an estimate for the univariate kernel which was proved in [9, Lemma 5.2].

Lemma 4.2. *Let $k_n^\delta(w^{(\xi,\eta)}; u)$, $u \in [-1, 1]$, denote the kernel for the univariate Cesàro (C, δ) means of the Jacobi expansion with respect to the weight function $w^{(\xi,\eta)}$ on $[-1, 1]$. Further, let $\xi, \eta > -1$ and $\delta \geq \xi + \eta + 2$. Then*

$$|k_n^\delta(w^{(\xi,\eta)}; t)| \leq cn^{-1}(1 - t - n^{-2})^{-(\xi+3/2)}.$$

It is well known that the Jacobi polynomials satisfy the following estimate ([10, (7.32.5) and (4.1.3)]).

Lemma 4.3. *For $\alpha, \beta > -1$ and $t \in [0, 1]$,*

$$|P_n^{(\alpha,\beta)}(t)| \leq cn^{-1/2}(1 - t + n^{-2})^{-(\alpha+1/2)/2}. \tag{4.1}$$

The estimate on $[-1, 0]$ follows from the fact that $P_n^{(\alpha,\beta)}(t) = (-1)^n P_n^{(\beta,\alpha)}(-t)$. In particular, for all $t \in [-1, 1]$, we have the estimate

$$|P_n^{(\alpha,\beta)}(t)| \leq cn^{-1/2}\left[(1 - t + n^{-2})^{-(\alpha+1/2)/2} + (1 + t + n^{-2})^{-(\beta+1/2)/2}\right]. \tag{4.2}$$

The central part of the proof is the following proposition.

Proposition 4.4. *If $\delta > \alpha + \beta + 1$ then*

$$\int_B \int_{-1}^1 |P_n^{(\alpha+\beta+\delta+\frac{1}{2},\beta)}(z(\mathbf{x}; t))| w^{(\alpha+\beta+\frac{1}{2},\beta)}(t) \,\mathrm{d}t\, W^{\alpha,\beta}(\mathbf{x}) \,\mathrm{d}\mathbf{x} \leq cn^{\delta-\alpha-\beta-3/2}.$$

Proof. Using the inequality (4.2), we see that it is sufficient to show that

$$J_1 := \int_B \int_{-1}^1 \frac{w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)}{(1 - z(\mathbf{x}; t) + n^{-2})^{\frac{\alpha+\beta+\delta+1}{2}}} \,\mathrm{d}t\, W^{\alpha,\beta}(\mathbf{x}) \,\mathrm{d}\mathbf{x} \leq cn^{\delta-\alpha-\beta-1}, \tag{4.3}$$

and

$$J_2 := \int_B \int_{-1}^1 \frac{w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)}{(1 + z(\mathbf{x}; t) + n^{-2})^{\frac{\beta+1/2}{2}}} \,\mathrm{d}t\, W^{\alpha,\beta}(\mathbf{x}) \,\mathrm{d}\mathbf{x} \leq cn^{\delta-\alpha-\beta-1}. \tag{4.4}$$

We start with J_1. Its estimate is divided into several cases, according to the decompositions $[-1, 1] = [-1, 0] \cup [0, 1]$ and

$$B = B_+ \cup B_-, \qquad B_+ := \{\mathbf{x} = (x_1, x_2) \in B : x_1 \geq 0\},$$
$$B_- := \{\mathbf{x} = (x_1, x_2) \in B : x_1 \leq 0\}.$$

To simplify notation, we further denote

$$\gamma := (\alpha + \beta + \delta + 1)/2$$

throughout this proof. The following basic identity can be easily verified,

$$1 - z(\mathbf{x}; t) = (1 - t^2)(1 - x_1) + \frac{1}{2}(1 - t)^2(1 - x_2). \tag{4.5}$$

Case 1. The integral over $\mathbf{x} \in B$ and $t \in [0, 1]$.

Since $t \geq 0$, by (4.5), $1 - z(\mathbf{x}; t) \geq (1-t)(1-x_1)$, and $w^{(\alpha+\beta+\frac{1}{2},\beta)}(t) \leq c(1-t)^{\alpha+\beta+\frac{1}{2}}$, so that

$$\int_0^1 \frac{w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)}{(1-z(\mathbf{x};t)+n^{-2})^\gamma} \mathrm{d}t \leq \int_0^1 \frac{(1-t)^{\alpha+\beta+\frac{1}{2}}}{[(1-t)(1-x_1)+n^{-2}]^\gamma} \mathrm{d}t =: f(x_1).$$

Since f depends on x_1 only, it readily follows that

$$\int_B f(x_1) W^{\alpha,\beta}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{-1}^1 f(x_1) \int_{x_1^2}^1 W^{\alpha,\beta}(\mathbf{x})\, \mathrm{d}x_2\, \mathrm{d}x_1 = c \int_{-1}^1 f(x_1)(1-x_1^2)^{\alpha+\beta}\, \mathrm{d}x_1,$$

where we used the fact that

$$\int_{x_1^2}^1 (1-x_2)^{\alpha-1/2}(x_2-x_1^2)^{\beta-1/2}\, \mathrm{d}x_2 = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\beta+1)}(1-x_1^2)^{\alpha+\beta}.$$

Thus, it follows that this part of the integral in J_1 is bounded by

$$\int_B \int_0^1 \cdots \leq c \int_{-1}^1 \int_0^1 \frac{(1-t)^{\alpha+\beta+\frac{1}{2}}}{[(1-t)(1-x_1)+n^{-2}]^\gamma}\, \mathrm{d}t\,(1-x_1^2)^{\alpha+\beta}\, \mathrm{d}x_1 := I_1^+ + I_1^-,$$

where we split the integral over $[-1, 1]$ as two integrals over $[0, 1]$ and $[-1, 0]$, respectively, and define I_1^+ and I_1^- accordingly. For I_1^+, we have

$$\begin{aligned} I_1^+ &= \int_0^1 \int_0^1 \frac{s^{\alpha+\beta+\frac{1}{2}}(1-x^2)^{\alpha+\beta}}{[s(1-x)+n^{-2}]^\gamma}\, \mathrm{d}s\, \mathrm{d}x = \int_0^1 \int_0^x \frac{u^{\alpha+\beta+\frac{1}{2}}(2+x)^{\alpha+\beta}}{(u+n^{-2})^\gamma}\, \mathrm{d}u\, x^{-\frac{3}{2}}\, \mathrm{d}x \\ &\leq \int_0^1 \frac{u^{\alpha+\beta+\frac{1}{2}}}{(u+n^{-2})^\gamma} \int_u^1 x^{-\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}u \leq c \int_0^1 \frac{u^{\alpha+\beta}}{(u+n^{-2})^\gamma}\, \mathrm{d}u \\ &= cn^{2\gamma-2\alpha-2\beta-2} \int_0^{n^2} \frac{v^{\alpha+\beta}}{(1+n^2 v)^\gamma}\, \mathrm{d}v \leq cn^{\delta-\alpha-\beta-1}, \end{aligned}$$

as the last integral is bounded if $\gamma > \alpha+\beta+1$. For I_1^-, we have $1 - x_1 \geq 1$, so that

$$I_1^- \leq c \int_0^1 \frac{(1-t)^{\alpha+\beta+\frac{1}{2}}}{[(1-t)+n^{-2}]^\gamma} \mathrm{d}t \int_{-1}^0 (1-x_1^2)^{\alpha+\beta}\, \mathrm{d}x_1 \leq c \int_0^1 \frac{u^{\alpha+\beta+1}}{(u+n^{-2})^\gamma}\, \mathrm{d}u,$$

which has the same bound as I_1^+ if we use $u^{\alpha+\beta+1} \leq u^{\alpha+\beta}$.

Case 2. The integral over $\mathbf{x} \in B_+$ and $t \in [-1, 0]$.

For $t \leq 0$, by (4.5), $1 - z(\mathbf{x}; t) \geq (1+t)(1-x_1) + \frac{1}{2}(1-x_2)$ and $w^{(\alpha+\beta+1,\beta)}(t) \leq c(1+t)^\beta$. Hence, this portion of the integral in J_1 is bounded by

$$I := c \int_{B_+} \int_{-1}^0 \frac{(1+t)^\beta}{\left[(1+t)(1-x_1)+(1-x_2)+n^{-2}\right]^\gamma}\, \mathrm{d}t\, W^{\alpha,\beta}(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Changing variables $u = (1+t)(1-x_1)$ shows that

$$\begin{aligned} I &= c \int_{B_+} \int_0^{1-x_1} \frac{u^\beta}{\left[u+(1-x_2)+n^{-2}\right]^\gamma}\, \mathrm{d}u \frac{W^{\alpha,\beta}(\mathbf{x})}{(1-x_1)^\beta}\, \mathrm{d}\mathbf{x} \\ &= c \int_0^1 \int_0^{\sqrt{x_2}} \int_0^{x_1} \frac{u^\beta}{\left[u+(1-x_2)+n^{-2}\right]^\gamma}\, \mathrm{d}u \frac{(x_2-x_1^2)^{\beta-\frac{1}{2}}}{(1-x_1)^{\beta+1}}\, \mathrm{d}x_1 (1-x_2)^{\alpha-\frac{1}{2}}\, \mathrm{d}x_2. \end{aligned}$$

Let us first consider the two inner integrals. Changing the order of integration shows that

$$\int_0^{\sqrt{x_2}} \int_0^{1-x_1} \mathrm{d}u \, \mathrm{d}x_1 = \int_0^{1-\sqrt{x_2}} \int_0^{\sqrt{x_2}} \mathrm{d}x_1 \, \mathrm{d}u + \int_{1-\sqrt{x_2}}^1 \int_0^{1-u} \mathrm{d}x_1 \, \mathrm{d}u.$$

Now, using $x_2 + \sqrt{x_1} \sim \sqrt{x_2}$ and $\beta + \frac{1}{2} > 0$, integration by parts once gives,

$$\begin{aligned}\int_0^{\sqrt{x_2}} \frac{(x_2 - x_1^2)^{\beta-\frac{1}{2}}}{(1-x_1)^{\beta+1}} \, \mathrm{d}x_1 &\le c x_2^{\beta-\frac{1}{2}} \int_0^{\sqrt{x_2}} \frac{(\sqrt{x_2} - x_1)^{\beta-\frac{1}{2}}}{(1-x_1)^{\beta+1}} \, \mathrm{d}x_1 \\ &= c \sqrt{x_2}^{\beta-\frac{1}{2}} \left[\frac{\sqrt{x_2}^{\beta+\frac{1}{2}}}{\beta+\frac{1}{2}} + \frac{\beta+1}{\beta+\frac{1}{2}} \int_0^{\sqrt{x_2}} \frac{1}{(1-x_1)^{\frac{3}{2}}} \, \mathrm{d}x_1 \right] \le c \frac{\sqrt{x_2}^{\beta-\frac{1}{2}}}{(1-x_2)^{\frac{1}{2}}}.\end{aligned}$$

Analogously, using $\sqrt{x_2} \le \sqrt{x_2} + x_1 \le 2$ and integration by parts once, we have the estimate

$$\begin{aligned}\int_0^{1-u} \frac{(x_2 - x_1^2)^{\beta-\frac{1}{2}}}{(1-x_1)^{\beta+1}} \, \mathrm{d}x_1 &\le c \max\{1, \sqrt{x_2}^{\beta-\frac{1}{2}}\} \int_0^{1-u} \frac{(\sqrt{x_2} - x_1)^{\beta-\frac{1}{2}}}{(1-x_1)^{\beta+1}} \, \mathrm{d}x_1 \\ &\le c \max\{1, \sqrt{x_2}^{\beta-\frac{1}{2}}\} \frac{1}{\sqrt{u}}.\end{aligned}$$

Adding the two parts together, we obtain that for a generic function $f(u)$,

$$\begin{aligned}&\int_0^{\sqrt{x_2}} \int_0^{1-x_1} f(u) \, \mathrm{d}u \frac{(x_2 - x_1^2)^{\beta-\frac{1}{2}}}{(1-x_1)^{\beta+1}} \, \mathrm{d}x_1 \\ &\quad \le c \max\{1, \sqrt{x_2}^{\beta-\frac{1}{2}}\} \int_0^1 f(u) \max\{(1-x_2)^{-\frac{1}{2}}, u^{-\frac{1}{2}}\} \, \mathrm{d}u.\end{aligned}$$

Consequently, we conclude that

$$I \le c \int_0^1 \int_0^1 \frac{u^\beta \max\{1, \sqrt{x_2}^{\beta-\frac{1}{2}}\}}{\left[u + (1-x_2) + n^{-2}\right]^\gamma} \, \mathrm{d}u \left[(1-x_2)^{-\frac{1}{2}} + u^{\frac{1}{2}} \right] (1-x_2)^{\alpha-\frac{1}{2}} \, \mathrm{d}x_2.$$

We note that the term $\max\{1, \sqrt{x_2}^{\beta-\frac{1}{2}}\}$, which matters only if $\beta < \frac{1}{2}$, is integrable as $\beta > -1/2$ and it plays a minor role. In fact, if we split the integral of x_2 as an integral over $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, then the part over $x_2 \in [0, \frac{1}{2}]$ is bounded as $u + (1-x_2) + n^{-2} \ge \frac{1}{2}$. Hence, we only need to estimate the sum

$$\int_0^1 \int_0^1 \frac{u^\beta x^{\alpha-1}}{(u + x + n^{-2})^\gamma} \, \mathrm{d}u \, \mathrm{d}x + \int_0^1 \int_0^1 \frac{u^{\beta-\frac{1}{2}} x^{\alpha-\frac{1}{2}}}{(u + x + n^{-2})^\gamma} \, \mathrm{d}u \, \mathrm{d}x := I_1 + I_2.$$

For I_1 we change variables $v = u + x$ and then exchange the order of the integrals,

$$\begin{aligned}I_1 = \int_0^1 \int_x^{x+1} \frac{(v-x)^\beta x^{\alpha-1}}{(v + n^{-2})^\gamma} \, \mathrm{d}v \, \mathrm{d}x &= \int_0^1 \frac{1}{(v + n^{-2})^\gamma} \int_0^v (v-x)^\beta x^{\alpha-1} \, \mathrm{d}x \, \mathrm{d}v \\ &\quad + \int_1^2 \frac{1}{(v + n^{-2})^\gamma} \int_{v-1}^1 (v-x)^\beta x^{\alpha-1} \, \mathrm{d}x \, \mathrm{d}v.\end{aligned}$$

The second term is bounded by a constant as $v \geq 1$, whereas the inner integral of the first term is a Beta integral and equals $v^{\beta+\alpha} B(\beta+1, \alpha)$, so that

$$I_1 \leq c \left(1 + \int_0^1 \frac{v^{\alpha+\beta}}{(v+n^{-2})^{\gamma}} \, \mathrm{d}v \right) \leq c n^{\delta-\alpha-\beta-1},$$

where the second inequality follows from the last step in the estimate for I_1^+ of Case 1. Notice that I_2 becomes I_1 if we replace (α, β) by $(\alpha - \frac{1}{2}, \beta + \frac{1}{2})$, so that I_2 has the same upper bound.

Case 3. The integral over $\mathbf{x} \in B_-$ and $t \in [-1, 0]$.

Just like in Case 2, this portion of the integral in J_1 is bounded by

$$I := c \int_{B_-} \int_{-1}^{0} \frac{(1+t)^{\beta}}{\left[(1+t)(1-x_1) + (1-x_2) + n^{-2}\right]^{\gamma}} \, \mathrm{d}t \, W^{\alpha,\beta}(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

For $\mathbf{x} \in B_-$, we have that $1 - x_1 \geq 1$ so that we can drop the factor $1 - x_1$ in the denominator. Changing variables $u = 1 + t$ then shows that

$$\begin{aligned} I &\leq c \int_{B_-} \int_{-1}^{0} \frac{u^{\beta}}{\left[u + (1-x_2) + n^{-2}\right]^{\gamma}} \, \mathrm{d}u (1-x_2)^{\alpha-\frac{1}{2}} (x_2 - x_1^2)^{\beta-\frac{1}{2}} \, \mathrm{d}\mathbf{x} \\ &\leq c \int_0^1 \int_0^1 \frac{u^{\beta} \, \mathrm{d}u}{(u + (1-x_2) + n^{-2})^{\gamma}} (1-x_2)^{\alpha-\frac{1}{2}} \int_{-\sqrt{x_2}}^{0} (x_2 - x_1^2)^{\beta-\frac{1}{2}} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &\leq c \int_0^1 \int_0^1 \frac{u^{\beta} (1-x_2)^{\alpha-1/2}}{(u + (1-x_2) + n^{-2})^{\gamma}} x_2^{\beta} \, \mathrm{d}u \, \mathrm{d}x_2 \leq c \int_0^1 \int_0^1 \frac{u^{\beta} x^{\alpha-1/2}}{(u + x + n^{-2})^{\gamma}} \, \mathrm{d}u \, \mathrm{d}x. \end{aligned}$$

Comparing to I_2 in Case 2 and using $u^{\beta} \leq u^{\beta-\frac{1}{2}}$, we see that I is bounded by $c n^{\delta-\alpha-\beta-1}$ as before.

Putting these cases together completes the proof of (4.3).

We now prove (4.4). A straightforward computation shows that

$$1 + z(\mathbf{x}; t) = \frac{1}{2} \left[\left(1 + t + (1-t)x_1\right)^2 + (1-t)^2 (x_2 - x_1^2) \right] \geq \frac{1}{2} (1-t)^2 (x_2 - x_1^2).$$

Using this inequality in J_2, we obtain that

$$\begin{aligned} J_2 &\leq c \int_B \int_{-1}^{1} \frac{(1-t)^{\alpha+\beta+\frac{1}{2}} (1+t)^{\beta}}{[(1-t)\sqrt{x_2 - x_1^2} + n^{-1}]^{\beta+\frac{1}{2}}} \, \mathrm{d}t (1-x_2)^{\alpha-\frac{1}{2}} (x_2 - x_1^2)^{\beta-\frac{1}{2}} \, \mathrm{d}\mathbf{x} \\ &\leq c \int_B \int_{-1}^{1} \frac{\left((1-t)\sqrt{x_2 - x_1^2}\right)^{\beta+\frac{1}{2}}}{[(1-t)\sqrt{x_2 - x_1^2} + n^{-1}]^{\beta+\frac{1}{2}}} (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\beta} \, \mathrm{d}t \\ &\quad \times (1 - x_2^2)^{\alpha-\frac{1}{2}} (x_2 - x_1^2)^{\frac{\beta-\frac{3}{2}}{2}} \, \mathrm{d}\mathbf{x} \\ &\leq c \int_B \int_{-1}^{1} (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\beta} \, \mathrm{d}t (1 - x_2^2)^{\alpha-\frac{1}{2}} (x_2 - x_1^2)^{\frac{\beta-\frac{3}{2}}{2}} \, \mathrm{d}\mathbf{x} \leq c. \end{aligned}$$

Hence, for $\delta > \alpha + \beta + 1$, J_2 is bounded and this completes the proof. □

Proof of Theorem 3.2. By Corollary 2.2, it is sufficient to show that

$$\|K_n^{\delta}\|_1 = \int_B |K_n^{\delta}(\mathbf{x})| W^{\alpha,\beta}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \leq c$$

under the condition that $\delta > \alpha + \beta + 1$. We set $J = \lfloor \alpha + 2\beta + \frac{3}{2} \rfloor$ in Lemma 4.1 and obtain that

$$|k_n^\delta(t)| \le \sum_{j=0}^{J} \left| b_j \left(\alpha + \beta + \frac{1}{2}, \beta, \delta, n \right) P_n^{(\alpha+\beta+\delta+j+\frac{3}{2},\beta)}(t) \right| + \sum_{j=J+1}^{\infty} \left| d_j \left(\alpha + \beta + \frac{1}{2}, \beta, \delta, n \right) k_n^{\delta+j}(t) \right|.$$

Using Corollary 3.1 together with Proposition 4.4, and taking into account the estimate for the coefficients given in Lemma 4.1, we obtain that

$$\|K_n^\delta\|_1 \le \frac{A_n^{\delta-1}}{A_n^\delta} \Bigg[cn + \sum_{j=J+1}^{\infty} \left| d_j \left(\alpha + \beta + \frac{1}{2}, \beta, \delta, n \right) \right| \times \int_B \int_{-1}^{1} |k_n^{\delta+j}(z(\mathbf{x}; t))| w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)\, \mathrm{d}t\, W^{\alpha,\beta}(\mathbf{x})\, \mathrm{d}\mathbf{x} \Bigg].$$

To estimate the second sum, we use Lemma 4.2. Thus, we have to derive an upper bound for the integral

$$I := n^{-1} \int_B \int_{-1}^{1} \frac{w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)}{(1 - z(\mathbf{x}; t) + n^{-2})^{\alpha+\beta+2}}\, \mathrm{d}t\, W^{\alpha,\beta}(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Setting $\gamma = \alpha + \beta + 2$, we already proved (cf. (4.3)) that

$$\int_B \int_{-1}^{1} \frac{w^{(\alpha+\beta+\frac{1}{2},\beta)}(t)}{(1 - z(\mathbf{x}; t) + n^{-2})^{\gamma}}\, \mathrm{d}t\, W^{\alpha,\beta}(\mathbf{x})\, \mathrm{d}\mathbf{x} \le cn^{2\gamma-2\alpha-2\beta-2} = cn^2,$$

provided that $\gamma > \alpha+\beta+1$. Hence, we obtain the bound $I \le cn$. From $\binom{n+\delta-1}{n} / \binom{n+\delta}{n} = \frac{\delta}{n+\delta}$ it follows that $\|K_n^\delta\|_1$ is bounded uniformly in n. □

Acknowledgment

The third author's work was partially supported by the NSF grant DMS-0604056.

References

[1] C.A. Agahanov, A method of constructing orthogonal polynomials of two variables for a certain class of weight functions, Vestnik Leningrad Univ. 20 (19) (1965) 5–10 (in Russian).
[2] F. Dai, Y. Xu, Cesàro means of orthogonal expansions in several variables, Const. Approx. 29 (1) (2009) 129–155.
[3] C.F. Dunkl, Y. Xu, Orthogonal Polynomials of Several Variables, in: Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, Cambridge, 2001.
[4] G. Gasper, Positive sums of the classical orthogonal polynomials, SIAM J. Math. Anal. 8 (1977) 423–447.
[5] T. Koornwinder, Jacobi polynomials, III. An analytic proof of the addition formula, SIAM J. Math. Anal. 6 (3) (1975) 533–543.
[6] T. Koornwinder, Two-variable analogues of the classical orthogonal polynomials, in: R.A. Askey (Ed.), Theory and Applications of Special Functions, Academic Press, New York, 1975, pp. 435–495.
[7] T. Koornwinder, A.L. Schwartz, Product formulas and associated hypergroups for orthogonal polynomials on the simplex and on a parabolic biangle, Constr. Approx. 13 (1997) 537–567.
[8] Zh. Li, Pointwise convergence of Fourier–Jacobi series, Approx. Theory Appl. (N.S.) 11 (4) (1995) 58–77.
[9] Zh. Li, Y. Xu, Summability of orthogonal expansions of several variables, J. Approx. Theory 122 (2003) 267–333.
[10] G. Szegö, Orthogonal Polynomials, 4th ed., in: American Mathematical Society Colloquium Publication, vol. 23, American Mathematical Society, Providence, RI, 1975.